Some Themed Problems

Math 110 Final Prep by Dan Sparks

I hope you find these problems interesting. I did! Two of them I borrowed from other GSI's (Mike Hartglass and Mohammad Safdari). These two problems, as well as one other (Problem 4), have already appeared in the worksheets. The rest are new and build upon the old ones.

Problem 1 (Mike Hartglass): Let T be an operator on a finite dimensional vector space. Show that T is diagonalizable if and only $T = a_1 P_1 + \cdots + a_k P_k$ where the $a_i \in \mathbf{F}$ are scalars and where the P_i are projections that commute with each other: i.e., $P_i P_j = P_j P_i$. □

Problem 2 (Mohammad Safdari): Let S, T be self-adjoint operators on a finite dimensional $\mathbf{R}$-inner product space [or let S, T be normal operators on a finite dimensional $\mathbf{C}$-inner product space]. Suppose also that $ST = TS$. Show that S, T are simultaneously orthogonally diagonalizable. That is, show there is an orthonormal basis consisting of vectors which are eigenvectors for both S and T. □

These were the original two problems which sparked my interest. First, let me point out a generalization of Problem 2 which has an important application in the theory of modular forms. For the curious, it is the space of newforms with the Petersson inner product, and the operators are the Hecke operators.

Problem 3 (New): Let S_i, for i in some index set I, be a collection of normal operators on a finite dimensional complex inner product space (or self-adjoint on a real inner product space). Suppose, for any $i, j \in I$, that $S_i S_j = S_j S_i$. Show that the S_i are simultaneously orthogonally diagonalizable. That is, show that there exists an orthonormal basis consisting of vectors which are eigenvectors for *every* operator S_i. [Suggestion: It would be entirely sufficient for this worksheet to suppose that I is a finite indexing set and use induction.] □

Recall a problem that I gave on an earlier worksheet.

Problem 4 (Dan Sparks): Let P be a projection on a finite dimensional inner product space. Prove that P is self-adjoint if and only if it is an orthogonal projection. □

Problem 5 (New): Prove the following orthogonal version of Problem 1. Let T be an operator on a finite dimensional inner product space. Then T is orthogonally diagonalizable (i.e., has an orthonormal eigenbasis) if and only if $T = a_1 P_1 + \cdots + a_k P_k$ where the a_i are scalars and the P_i are *orthogonal* projections such that $P_i P_j = P_j P_i$ for all i, j. [Suggestion: There's an easy proof using the preview two exercises.] □

Problem 6 (New): Let T be a diagonalizable operator on a finite dimensional vector space V. Suppose that U is a T-invariant subspace. Show that $T|_U$ is diagonalizable. □

Problem 7 (New): Let S_i, for i in some index set I, be a collection of diagonalizable operators on a finite dimensional vector space V. Suppose, for any $i, j \in I$, that $S_i S_j = S_j S_i$. Show that the S_i are simultaneously diagonalizable. [Suggestion: If necessary, first do the case of just two operators S, T. Again, suppose that I is finite, and try induction.] □

Problem 8 (New): Give an easy proof of Problem 1 using Problem 7. □

Note that Problem 3 and Problem 7 are analogous to each other, in the same way that Problem 1 and Problem 5 are analogous to each other. Notice especially that neither the plain version nor the orthogonal version is more general than the other. The non-orthogonal versions apply in more contexts, but the orthogonal versions give sharper results. On the other hand, I do think that Problem 3 and Problem 7 are more general/powerful than Problem 1 and Problem 5, since you can give easy proofs of 1 and 5 using 7 and 3 respectively.